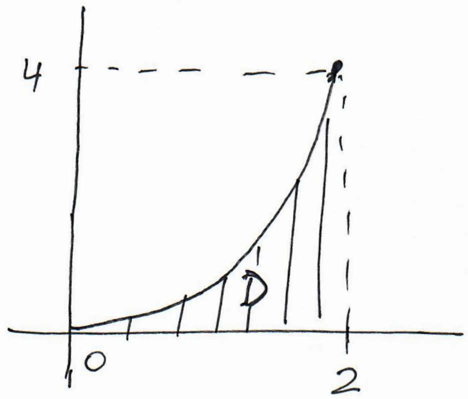
15.1 Definite Integral

Given $y = f(x) = x^2$, $\quad 0 \leq x \leq 2$

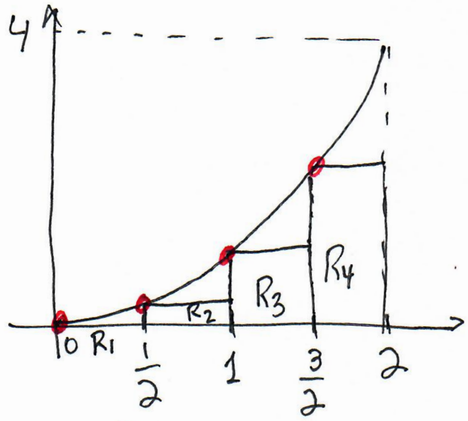
Compute the area under the curve.



$$\text{Area of } D = \int_0^2 f(x)\,dx = \int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

$$\simeq \underline{\underline{2.66}}$$

This area can be approximated by Riemann Sums. (Lower Rectangles)



$$\text{Area of } D \simeq \text{Area}(R_1) + \text{Area}(R_2) + \text{Area}(R_3) + \text{Area}(R_4).$$

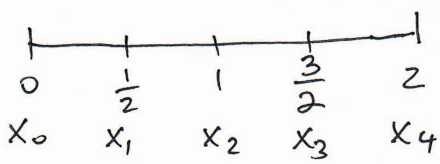
$$= f(0)\tfrac{1}{2} + f(1/2)\tfrac{1}{2} + f(1)\tfrac{1}{2} + f\left(\tfrac{3}{2}\right)\tfrac{1}{2}$$

$$= 0\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2\tfrac{1}{2} + (1)^2\tfrac{1}{2} + \left(\tfrac{3}{2}\right)^2\tfrac{1}{2}$$

$$= 0 + \frac{1}{8} + \frac{1}{2} + \frac{9}{8} = \frac{14}{8}$$

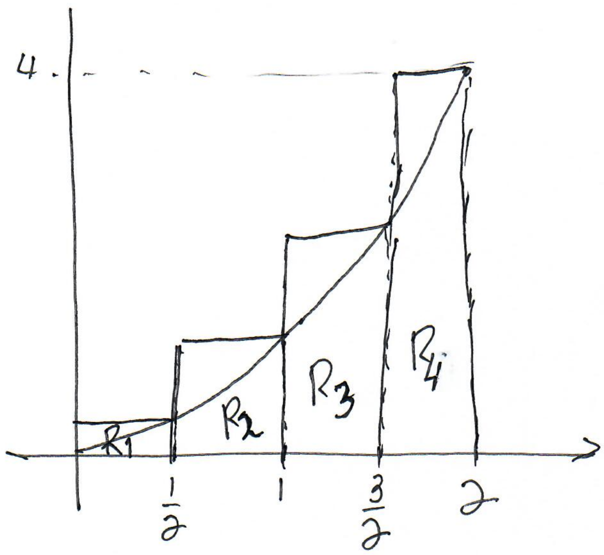
$$\simeq \underline{\underline{1.75}}.$$

Uniform partition:



$x_i - x_{i-1} = \frac{1}{2}$, $\quad i = 1, 2, 3, 4$

Using Upper rectangles:



$$\text{Area of } D \simeq \text{Area}(R_1) + \dots + \text{Area}(R_4)$$

$$= f(1/2)\tfrac{1}{2} + f(1)\tfrac{1}{2} + f(\tfrac{3}{2})\tfrac{1}{2} + f(2)\tfrac{1}{2}$$

$$= \left(\tfrac{1}{2}\right)^2\tfrac{1}{2} + 1^2\tfrac{1}{2} + \left(\tfrac{3}{2}\right)^2\tfrac{1}{2} + 2^2\tfrac{1}{2}$$

$$= \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 2 = \frac{15}{4}$$

$$\simeq 3.75$$

Area (Lower. rectang) $\leq$ Area $\leq$ Area (upp. rectang)

$$1.75 \leq 2.66 \leq 3.75$$

Clearly, as we refine the uniform partition

$\Delta x = x_i - x_{i-1}$ is smaller

the approx. improves.

<u>Definition</u>

The <u>definite integral</u> of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x, \quad \overset{\text{arbitrary point}}{x_i^* \in [x_{i-1}, x_i]}$$

provided that this limit exists.

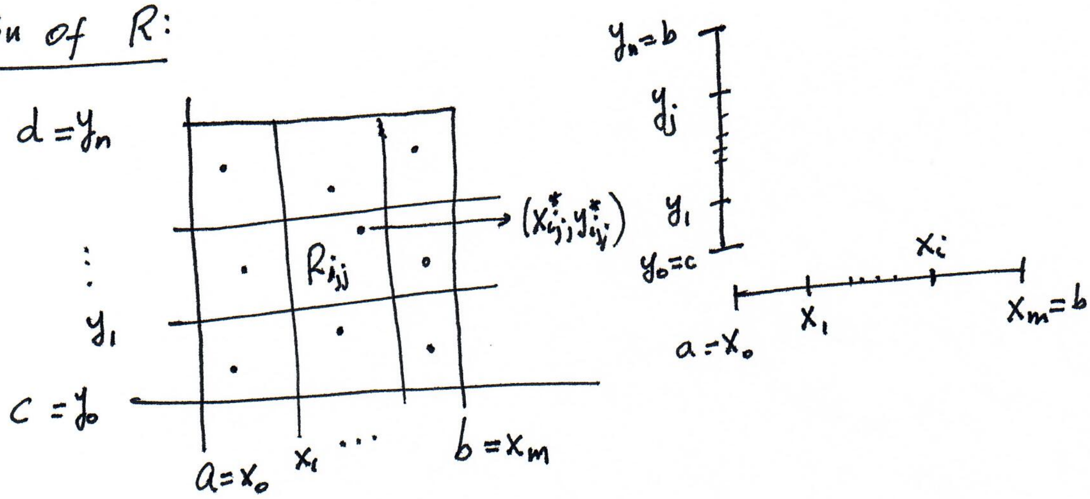
<u>Notice</u>: For a uniform partition $\Delta x \to 0$ is equivalent to $n \to \infty$.

Riemann Sums:

$f(x,y)$ defined on a closed rectangle

$$R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, \quad c \le y \le d\} = [a,b] \times [c,d]$$

Partition of R:



Uniform in both directions:

$\Delta x = x_i - x_{i-1}, \qquad i = 1, 2, \dots m$

$\Delta y = y_j - y_{j-1}, \qquad j = 1, 2, \dots, n$

$R_{ij} \equiv \{(x,y) \mid x_{i-1} \le x \le x_i, \quad y_{j-1} \le y \le y_j\}$

the point (x_{ij}^*, y_{ij}^*) is called a sample point. This is an arbitrary point in R_{ij}.

The double Riemann Sums are defined as

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

where $\Delta A = \Delta R_{ij} = \Delta x \, \Delta y$

Def - f defined on $R = [a,b] \times [c,d]$

(f is not necessarily continuous or positive.)

Then, the double integral of f over the rectangle R is given by

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

if this limit exists.

Remark: If conts and $f \geq 0$ on R

$$\text{Volume of } S = \iint_R f(x,y)\,dA$$

Where $S = \{(x,y,f(x,y)) \in \mathbb{R}^3 \mid (x,y) \in R\}$

15.1 Iterated Integrals.

Theorem.- (Fubini's Thm.)

If f is conts on $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy\right]dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right]dy$$

Rmk: Theorem is also true if f bounded in R, f discont. only on a finite number of smooth curves, and iterated integrals exist.

The integrals $\int_a^b \left[\int_c^d f(x,y)\,dy\right]dx$ and $\int_c^d \left[\int_a^b f(x,y)\,dx\right]dy$

are called iterated integrals.

AVERAGE VALUE.

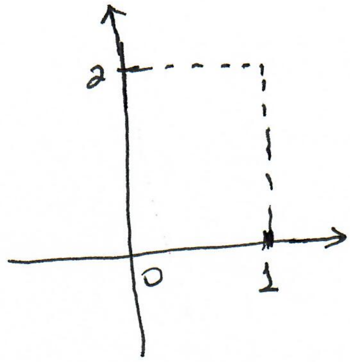
$$f_{avg} = \frac{1}{A(R)} \iint_R f(x,y)\,dA.$$

15.2

Example

Compute $$\iint_R xy\, e^{x^2 y}\, dA$$

where $R = \underset{x}{[0,1]} \times \underset{y}{[0,2]}$



Answ: Two possibilities using Iterated integrals:

Case 1
$$\int_{\substack{0\\x}}^{1} \int_{\substack{0\\y}}^{2} xy\, e^{x^2 y}\, dy\, dx$$

$$= \int_{\substack{0\\x}}^{1} \left(\int_{\substack{0\\y}}^{2} y x\, e^{x^2 y}\, dy \right) dx$$

Integration in y requires integration by parts.

Similar to $\int y e^{y} dy$

Case 2
$$\int_{\substack{0\\y}}^{2} \int_{\substack{0\\x}}^{1} xy\, e^{x^2 y}\, dx\, dy$$

$$= \int_{0}^{2} \left(\int_{0}^{1} xy\, e^{x^2 y}\, dx \right) dy$$

Case 2 (cont.)

Substitution
$u = x^2 y, \quad du = 2xy\,dx$

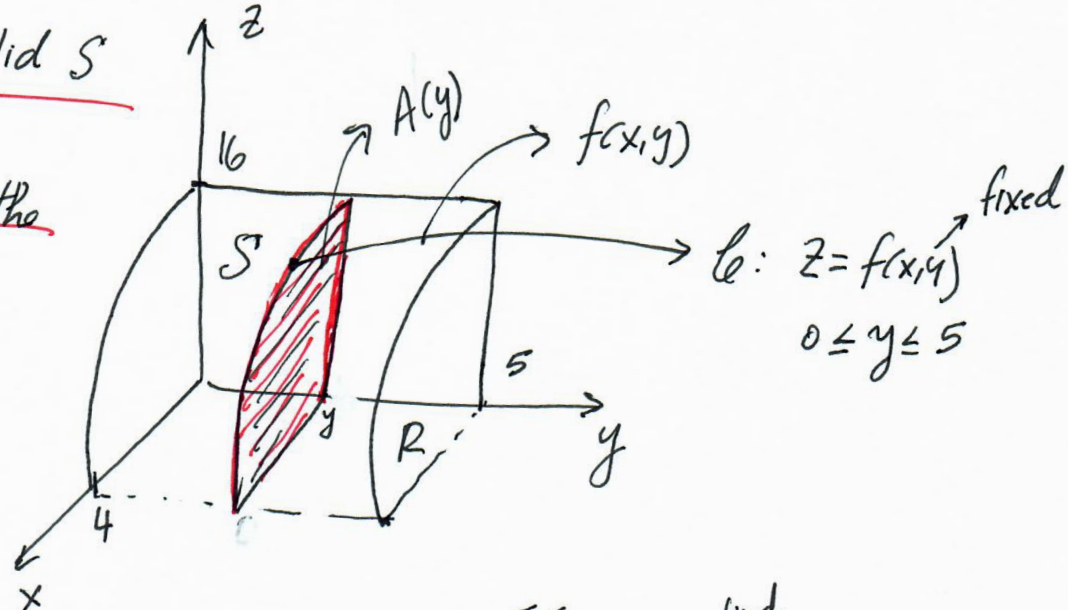
$$\int_0^2 \left(\int_0^1 xy\, e^{x^2 y}\, dx \right) dy =$$

$$= \frac{1}{2} \int_0^2 \left(\int_0^1 2xy\, e^{x^2 y}\, dx \right) dy = \frac{1}{2} \int_0^2 \left(\int_0^y e^u\, du \right) dy$$

$$= \frac{1}{2} \int_0^2 e^u \Big|_0^y\, dy = \frac{1}{2} \int_0^2 (e^y - 1)\, dy =$$

$$= \frac{1}{2} \left(e^y - y \Big|_0^2 \right) = \frac{1}{2} \left(e^2 - 2 - 1 \right) = \frac{e^2 - 3}{2}.$$

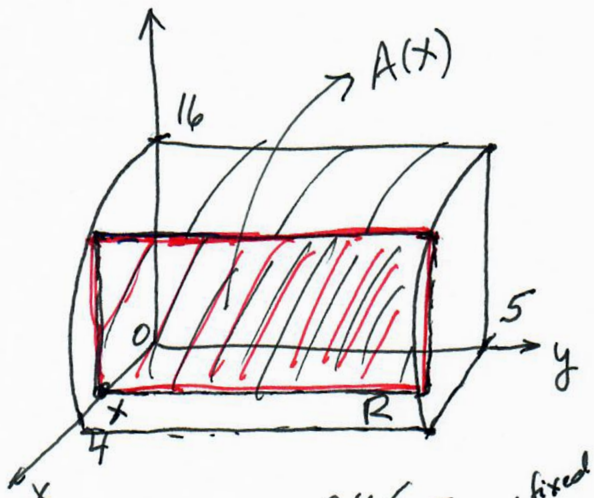
Volume of solid S enclosed by R under the graph of f.



$$\text{Vol. of Solid} = \int_0^5 A(y)\,dy = \int_0^5 \left(\int_0^4 f(x, \overset{\text{fixed}}{y})\,dx \right) dy$$

$$= \int_0^5 \int_0^4 f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$$

Alternative:



$$\text{Vol. of solid} = \int_0^4 A(x)\,dx = \int_0^4 \left(\int_0^5 f(\overset{x\text{ fixed}}{x},y)\,dy \right) dx$$

$$= \int_0^4 \int_0^5 f(x,y)\,dy\,dx = \iint_R f(x,y)\,dA.$$

15.1

#32 $\displaystyle\iint_R \frac{x}{1+xy}\,dA \qquad R=[0,1]\times[0,1]$

$$\int_0^1\left(\int_0^1 \frac{x}{1+xy}\,dy\right)dx = \int_0^1 \ln(1+xy)\Big|_0^1 dx$$

$$= \int_0^1 [\ln(1+x) - \overset{=0}{\ln(1+0)}]\,dx$$

$$= \int_0^1 \ln(1+x)\,dx = \int_1^2 \ln(u)\,du =$$

$u = 1+x, \quad du = dx$

$$= u\ln u - u\Big|_1^2 = 2\ln(2) - 2 - (\overset{=0}{\ln(1)} - 1)$$

$$= 2\ln(2) - 2 + 1 = \underline{\underline{2\ln(2) - 1}}$$

$$\int \ln u\,du \overset{\text{by parts}}{=} \int \underset{g'}{1}\,\underset{f}{\ln u}\,du = fg - \int f'g$$

$$= u\ln u - \int \frac{1}{\cancel{u}}\cdot\cancel{u}\,du = \underline{\underline{u\ln u - u}}$$

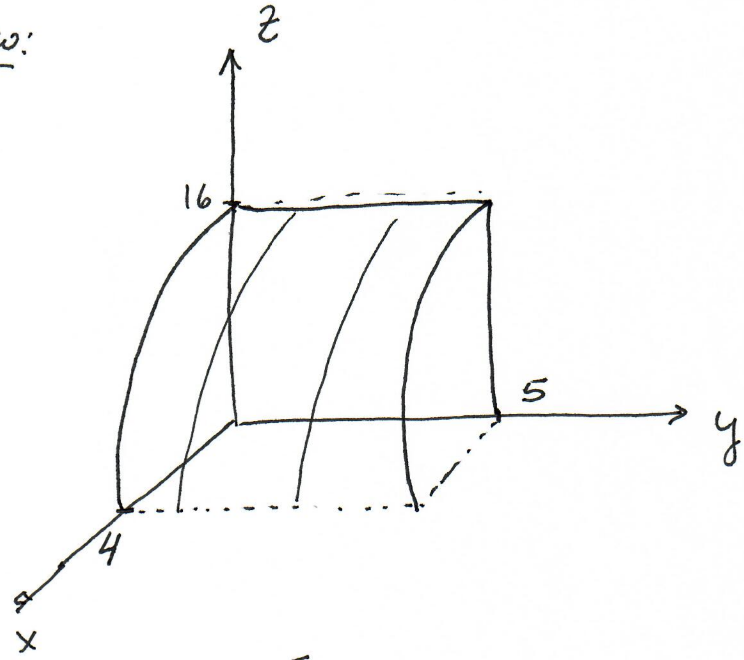
15.2

#30) Find the volume of the solid in the first octant bounded by the cylinder

$$z = 16 - x^2 \quad \text{and}$$

plane : $y = 5$

Answ:



Case 1 : $\displaystyle\int_0^5 \int_0^4 (-x^2+16)\,dx\,dy =$

$$\int_0^5 \left(\int_0^4 (-x^2+16)\,dx \right) dy = \int_0^5 -\frac{x^3}{3} + 16x \Big|_0^4 \, dy = \int_0^5 \left(-\frac{64}{3} + 64 \right) dy$$

$$= \int_0^5 \frac{128}{3}\,dy = 5\left(\frac{128}{3}\right) = \frac{640}{3}$$

Remark: Come back to this problem later for other possibilities.